ALGEBRA

Fabio

- Polinomi
- Disuguaglianze tra medie
  (corollario: come farsi alzare la media dei voti dai professori di materie umanistiche senza fare altre verifiche)

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

$x^2 - 3x + 2$ $\qquad a_n \neq 0$

$$\begin{array}{r|l} 13 & 3 \\ \hline 1 & 4 \end{array}$$

$13 = 3 \cdot 4 + 1$

$$\begin{array}{l|l} x^5 + 3x^2 + 1 & x^2 + 2 \\ \cline{2-2} x^5 + 2x^3 & x^3 - 2x + 3 \\ \cline{1-1} -2x^3 + 3x^2 + 1 & \\ -2x^3 - 4x & \\ \cline{1-1} 3x^2 + 4x + 1 & \\ 3x^2 + 6 & \\ \cline{1-1} 4x - 5 & \end{array}$$

$x^5 + 3x^2 + 1 = (x^2+2)(x^3 - 2x + 3) + (4x - 5)$

Teorema di Ruffini

$P(x)$ polinomio. Divido per $(x - a)$.
Allora il resto vale $P(a)$

$$P(x) = (x-a)\,Q(x) + r$$

$$P(a) = r$$

a è radice del polinomio se $P(a) = 0$

$$P(x) = (x-a)\,Q(x)$$

$p(2)=2\ ,\ p(3)=2\ ,\ p(5)=2 \quad \leftarrow$

$G(x)=P(x)-2 \rightsquigarrow G(2)=G(3)=G(5)=0$

$P(x)-2=G(x)=(x-2)(x-3)(x-5)Q(x)$

$P(x)=(x-2)(x-3)(x-5)Q(x)+2$

ES: $p(x)$, $q(x)$ di grado ≤ 3, distinti, a coefficienti interi.

$p(1)=q(1)\ ,\ p(2)=q(2)\ ,\ p(3)=q(3)$

$p(-1)=-q(-1),\ p(-2)=-q(-2)\ ,\ p(-3)=-q(-3)$

Quanto vale al minimo $[p(0)]^2+[q(0)]^2$?

$p(x)-q(x)=(x-1)(x-2)(x-3)\,a \qquad *$

$p(x)+q(x)=(x+1)(x+2)(x+3)\,b$

$p(0)-q(0)=-6a\ ,\ p(0)+q(0)=6b$

$p(0)^2+q(0)^2-2p(0)q(0)=36a^2$

$p(0)^2+q(0)^2+2p(0)q(0)=36b^2$

$[p(0)]^2+[q(0)]^2=18(\underset{\neq 0}{a^2+b^2})$

1	0
2	0 $\leftarrow 4$
1	1 $\leftarrow 2$

$p(x)+q(x)=0$

$2p(x)=(x-1)(x-2)(x-3)\frac{1}{2}$

$(x-1)(x-2)(x-3)=x^3-6x^2+11x-6 \quad =p-q$

$x^3+6x^2+11x+6 \quad =p+q$

$p=x^3+11x \qquad q=6x^2+6$

$\boxed{36}$

Es. $P(x)$ di grado 999.

$$P(1)=1 \quad P(2)=\frac{1}{2} \quad P(3)=\frac{1}{3} \quad \dots \quad P(1000)=\frac{1}{1000}$$

Quanto vale $P(1001)$?

$$Q(x)=xP(x)-1 \qquad Q(0)=-1$$

$$Q(x)=a(x-1)(x-2)(x-3)\cdot\ldots\cdot(x-1000) \leftarrow$$

$$-1=Q(0)=a(-1)(-2)\cdot\ldots\cdot(-1000)=a\cdot 1000!$$

$$a=-\frac{1}{1000!}$$

$$Q(1001)=1001\cdot P(1001)-1$$
$$\overset{!}{=}-\frac{1}{1000!}(1000)\cancel{(999)}(998)\cdot\ldots\cdot 2\cdot 1=-1$$

Quindi $P(1001)=0$.

Disuguaglianze tra medie

$a_1, a_2, a_3, \dots, a_n > 0$

$$AM=\frac{a_1+a_2+a_3+\ldots+a_n}{n}$$

$$QM=\sqrt{\frac{a_1^2+a_2^2+a_3^2+\ldots+a_n^2}{n}}$$

$$GM=\sqrt[n]{a_1a_2a_3a_4\ldots a_n}$$

$$HM = \frac{m}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_m}}$$

$$M_p = \left(\frac{a_1^p + a_2^p + \dots + a_m^p}{m} \right)^{1/p}$$

$1 \quad 2$

$\frac{3}{2} \qquad \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$

$$\min\{a_1, \dots, a_m\} \leq HM \leq GM \leq AM \leq QM \leq \max\{a_1, \dots, a_m\}$$

Es: $a, b, c, d > 0$ tali che $abcd = 1$.

Quanto vale al minimo $abc + abd + acd + bcd$?

$$\underbrace{\sqrt[4]{a^3 b^3 c^3 d^3}} = 1$$

$$1 = GM \leq AM = \frac{abc + abd + acd + bcd}{4}$$

$a = b = c = d = 1 \implies$ Risposta: 4

Es:

$a + 3b + 9c = 171$

$$\sqrt[3]{a \cdot 3b \cdot 9c} \leq \frac{171}{3} = 57$$

$$\underbrace{\sqrt[3]{27}}_{=3} \sqrt[3]{abc} \leq 57 \qquad abc \leq 19^3 = 6859$$

$$a = \frac{171}{3}, \ b = \frac{171}{9}, \ c = \frac{171}{27}$$

Esercizi

1. $p(x) = \left(x^{39} - x^4 + 1\right)^{2019}$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

$$p(1) = (1-1+1)^{2019} = 1 \qquad p(-1) = (-1-1+1)^{2019} = -1$$

$$p(-1) = a_0 - a_1 + a_2 - a_3 + \dots$$

$$\left(p(1) - p(-1)\right)\cdot\frac{1}{2} = \left(1 - (-1)\right)\cdot\frac{1}{2} = 1$$

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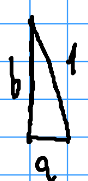
2. $p(1) = 1 \quad p(2) = 2 \quad p(3) = 3$

$$q(x) = p(x) - x = (x-1)(x-2)(x-3)\overset{\overset{a}{\shortparallel}}{\underbrace{s(x)}}$$

$$p(0) = -6\,s(0) = -6a \rightsquigarrow p(0)$$ può essere un qualunque multiplo di 6.

3.



$$a^2 + b^2 = 1$$

$$\sqrt{ab} \le \sqrt{\frac{a^2+b^2}{2}} \Rightarrow ab \le \frac{1}{2} \Rightarrow \frac{ab}{2} \le \frac{1}{4}$$

$$GM \le QM$$

$$a = b = \frac{1}{\sqrt{2}} \Longrightarrow \text{Area} = \frac{1}{4}$$

4.(*) $GM \le AM$ $\qquad \underbrace{\frac{a}{b}} + \underbrace{\sqrt{\frac{b}{a}}}$

$$\frac{a}{b} + \sqrt{\frac{b}{a}} = \underbrace{\frac{a}{b}} + \underbrace{\frac{1}{2}\sqrt{\frac{b}{a}}} + \underbrace{\frac{1}{2}\sqrt{\frac{b}{a}}}$$

$$GM = \sqrt[3]{\frac{a}{b}\cdot\frac{1}{2}\sqrt{\frac{b}{a}}\cdot\frac{1}{2}\sqrt{\frac{b}{a}}} = \frac{1}{\sqrt[3]{4}} \le \frac{\frac{a}{b} + \sqrt{\frac{b}{a}}}{3}$$

$$\frac{a}{b}+\sqrt{\frac{b}{a}} \geq \frac{3}{\sqrt[3]{4}}$$

$$\left(\frac{a}{b}=\frac{1}{2}\sqrt{\frac{b}{a}} \right.$$

$$x=\frac{1}{2}\sqrt{\frac{1}{x}} \qquad 4x^2=\frac{1}{x}$$

$$\left. 4x^3=1 \rightsquigarrow \frac{a}{b}=\frac{1}{\sqrt[3]{4}} \right)$$